Journal of Statistical Physics, Vol. 100, Nos. 5/6, 2000

Ising Models on Hyperbolic Graphs II

C. Chris Wu[1]

Received October 12, 1999; final May 20, 2000

We consider Ising models with ferromagnetic interactions and zero external magnetic field on the hyperbolic graph $\mathscr{H}(v, f)$, where v is the number of neighbors of each vertex and f is the number of sides of each face. Let T_c be the critical temperature and $T'_c = \sup\{T \leqslant T_c : \nu^f = (\nu^+ + \nu^-)/2\}$, where ν^f is the free boundary condition (b.c.) Gibbs state, ν^+ is the plus b.c. Gibbs state and ν^- is the minus b.c. Gibbs state. We prove that if the hyperbolic graph is self-dual (i.e., $v = f$) or if v is sufficiently large (how large depends on f, e.g., $v \geqslant 35$ suffices for any $f \geqslant 3$ and $v \geqslant 17$ suffices for any $f \geqslant 17$) then $0 < T'_c < T_c$, in contrast with that $T'_c = T_c$ for Ising models on the hypercubic lattice $\mathbf{Z}^d$ with $d \geqslant 2$, a result due to Lebowitz.[22] While whenever $T < T'_c$, $\nu^f = (\nu^+ + \nu^-)/2$. The last result is an improvement in comparison with the analogous statement in refs. 28 and 33, in which it was only proved that $\nu^f = (\nu^+ + \nu^-)/2$ when $T \ll T'_c$ and it remains to show in both papers that $\nu^f = (\nu^+ + \nu^-)/2$ whenever $T < T'_c$. Therefore T'_c and T_c divide $[0, \infty]$ into three intervals: $[0, T'_c)$, (T'_c, T_c), and $(T_c, \infty]$ in which $\nu^+ \neq \nu^-$ but $\nu^f = (\nu^+ + \nu^-)/2$, $\nu^+ \neq \nu^-$ and $\nu^f \neq (\nu^+ + \nu^-)/2$, and $\nu^+ = \nu^-$, respectively.

KEY WORDS: Ising models; percolation; hyperbolic graphs.

1. INTRODUCTION

The study of Ising models and percolation has been mainly focused on the finite-dimensional lattice $\mathbf{Z}^d$ during the past half a century. However, in recent years increasing attention has been dedicated to the study of these models on other graphs. A sample of these efforts can be found in the papers[3–7, 15, 17, 19–21, 25, 28–30, 32, 33] and the references therein. The graphs considered in these papers have the property that they are infinite dimensional in the sense that the graphs grow exponentially fast, but they are not

[1] Department of Mathematics, Penn State University, Beaver Campus, 100 University Drive, Monaca, Pennsylvania 15061.

0022-4715/00/0900-0893$18.00/0

trees, i.e., they have loops. Series and Sinai[32] studied Ising models on certain hyperbolic graphs and proved that in low temperature there are uncountably many mutually singular Gibbs states which, they believed, are extremal. Using the methods of high- and low-temperature expansions Rietman *et al.*[29] evaluated the critical temperature and the critical exponents β and γ, associated with the magnetization and the susceptibility respectively. Not surprisingly their numerical results show that β and γ take their mean-field values 1/2 and 1. What is surprising is that their numerical results indicate that for self-dual hyperbolic graphs there may be a second critical temperature; but the low-temperature expansions of the magnetization and susceptibility do not show this singularity. In refs. 28 and 33 it is proved that Ising models on certain nonamenable graphs exhibit multiple phase transitions; the graph considered in ref. 28 is transitive but not planar and the one in ref. 33 is planar but not transitive.

The motivation for this trend of studying the models on general graphs is at least twofold. On one hand, it is clear from the results in the papers quoted above that on graphs different from the commonly studied square lattices and homogeneous trees there is new mathematical structure to be found. On the other hand the techniques and results obtained from the study of the models on more general graphs in some instances have led to results which were not yet available for the most studied particular graphs.

In this paper we study Ising models on hyperbolic graphs considered in ref. 29. These graphs are planar and transitive. We will show that the models have a second phase transition in the sense to be explained below. The hyperbolic graphs can be briefly described as follows (for their detailed constructions see Section 2 of ref. 29). Each vertex of the graph has the same number of neighbors. The graphs, with equilateral faces, can be embedded in a two-dimensional space with constant curvature. The space is the sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{R}^2$ or the hyperbolic plane $\mathbf{H}^2$ respectively when the curvature is positive, zero or negative respectively. The graphs can be characterized by two integers, both $\geqslant 3$: v, the number of neighbors of each vertex; and f, the number of sides of each face, and denoted as (v, f). They can be embedded in $\mathbf{S}^2$, $\mathbf{R}^2$ or $\mathbf{H}^2$ respectively when the quantity $(v-2)(f-2)$ is smaller than, equal to or larger than four respectively. The (five) graphs which can be embedded in the sphere have only finite number of vertices (e.g., the (3, 3) lattice is a tetrahedron). On $\mathbf{R}^2$ there are three (infinite) graphs: triangular (6, 3), square (4, 4) and hexagonal (3, 6). Ising systems on these lattices have received intensive study. Among the celebrated results is that for Ising models on $\mathbf{Z}^2$, all Gibbs states are translation invariant and are convex combinations of the (+) boundary condition (b.c.) state ν^+ and the (−) b.c. state ν^-, proved independently by Aizenman[1] and Higuchi.[16] When embedded in the hyperbolic plane, there are infinitely many choices of (v, f). We denote these graphs by $\mathscr{H}(v, f)$ when $(v-2)(f-2)>4$. We will occasionally write $\mathscr{H}$ for $\mathscr{H}(v, f)$ for simplicity. As in the case of the three graphs embedded in $\mathbf{R}^2$, the dual graph of $\mathscr{H}(v, f)$ is the graph $\mathscr{H}(f, v)$. The hyperbolic graph $\mathscr{H}(5, 5)$, which is self-dual, is shown in Fig. 1 as an example.

Before we state the results for Ising models on the hyperbolic graphs, we first recall some fundamental results for Ising models on $\mathbf{Z}^d$. On $\mathbf{Z}^2$, as mentioned above, there is no non-translation-invariant Gibbs state and ν^+ and ν^- are the only extremal Gibbs states. While on $\mathbf{Z}^d$ with $d \geqslant 3$, Dobrushin[9] proved that for (very) low temperature there exist non-translation-invariant Gibbs states, which of course are not combinations of ν^+ and ν^-. Meanwhile, it is expected, although not completely proved, that ν^+ and ν^- are the only translation invariant extremal Gibbs states and any other translation invariant Gibbs state is a combination of them—this has been proved for all low temperatures by Gallavotti and Miracle-Sole[13] and for all but countably many temperatures below the critical temperature T_c by Lebowitz,[22] where T_c is defined, as usual, as the supremum of the

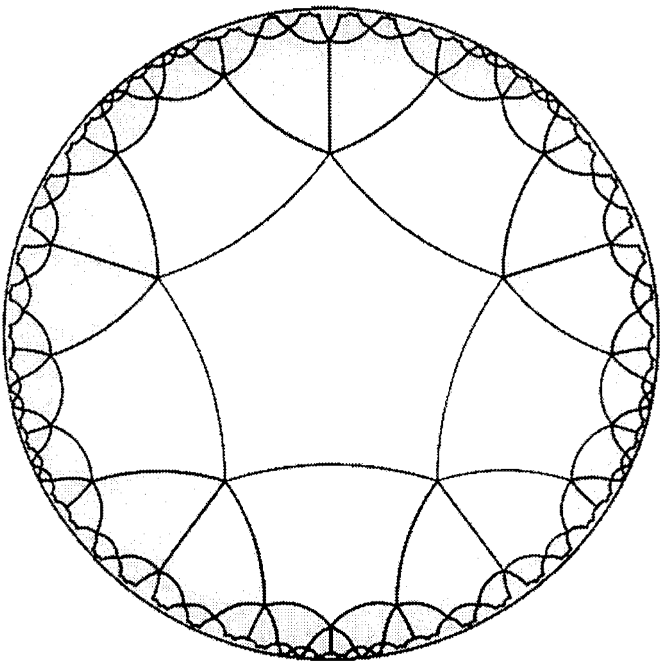

Fig. 1. An example of the hyperbolic graph $\mathscr{H}(5, 5)$ drawn on the unit disk.

temperatures at which the spontaneous magnetization at the origin is positive. In particular, the Gibbs state with free b.c., ν^f, which is translation invariant, is equal to $(\nu^+ + \nu^-)/2$ for all but countably many temperatures.

Returning to Ising models on the hyperbolic graphs, it is known by the work of Series and Sinai[32] that there are uncountably many extremal Gibbs states for T (well) below T_c, in contrast with the widely believed view (but not rigorously proved) that the total number of extremal Gibbs states (translation invariant and non-translation-invariant) is countable for Ising models on $\mathbf{Z}^d$ with $d \geqslant 3$. In this paper we prove that if the hyperbolic graph $\mathscr{H}(v, f)$ is self-dual, i.e., $v = f$, or if v is sufficiently large (how large depends on f, e.g., $v \geqslant 35$ suffices for any $f \geqslant 3$ and $v \geqslant 17$ suffices for any $f \geqslant 17$), then there exists a second critical temperature $T'_c < T_c$ such that for T between T'_c and T_c, $\nu^f \neq (\nu^+ + \nu^-)/2$—hence ν^+ and ν^- are not the only translation invariant extremal Gibbs states (in contrast with that ν^+ and ν^- are the only translation invariant extremal Gibbs states for Ising models on $\mathbf{Z}^d$ with $d \geqslant 3$); while whenever $T < T'_c$, $\nu^f = (\nu^+ + \nu^-)/2$. We remark that similar results were obtained in ref. 28 for a transitive but non-planar graph and in ref. 33 for a planar but non-transitive graph. However in both papers it was only shown that $\nu^f = (\nu^+ + \nu^-)/2$ for $T \ll T'_c$, i.e., for T sufficiently below T'_c. It remains to show in both papers that $\nu^f = (\nu^+ + \nu^-)/2$ whenever $T < T'_c$.

We expect that the results are true on any $\mathscr{H}(v, f)$ as long as $(v-2)(f-2) > 4$, but at the present we need to assume that $\mathscr{H}(v, f)$ is self-dual or v is sufficiently large to guarantee that T'_c is strictly less than T_c. The argument in this paper also applies to q-state Potts models. But it requires v to be larger to guarantee that T'_c is strictly less than T_c when $q > 2$. We conclude the introduction with some open problems: For T between T'_c and T_c how many translation invariant extremal Gibbs states are there and what are they? Is ν^f extremal? For T below T'_c are ν^+ and ν^- the only translation invariant extremal Gibbs states?

2. STATEMENT OF RESULTS

Let $V(\mathscr{H})$ and $E(\mathscr{H})$ be the vertex (site) set and edge (bond) set of $\mathscr{H}$ respectively. The ferromagnetic Ising model on the graph $\mathscr{H}$ is described by the spin random variables $\{\sigma_x : x \in V(\mathscr{H})\}$. Each σ_x takes on the values ± 1. The interaction between the spins is described by the Hamiltonian:

$$H = -\tfrac{1}{2}\beta \sum_{\{x,\, y\}} \sigma_x \sigma_y$$

in which the sum is over nearest neighbor bonds $\{x, y\}$, and $\beta = 1/T \geqslant 0$ is the inverse temperature. Let ν_β^+ (respectively ν_β^- or ν_β^f) denote the Gibbs state associated with the Hamiltonian H with $(+)$ (respectively $(-)$, or free) b.c. Let $\langle \cdot \rangle_{*,\beta}$ denote the expectation with respect to ν_β^*, where $* = +,\ -$, or f. When it does not cause confusion, we will drop the subscript β in ν_β^* and $\langle \cdot \rangle_{*,\beta}$. Let o be a distinguished site of $V(\mathscr{H})$ called the origin. The infinite volume quantities of primary interest to us are the spontaneous magnetization

$$M(\beta) \equiv M_o(\beta) = \langle \sigma_o \rangle_{+,\beta} \tag{2.1}$$

and the two-point correlation function $\langle \sigma_x \sigma_y \rangle_{*,\beta}$ with $*$ b.c.

Before stating the results of Ising models on $\mathscr{H}(v, f)$, we first review a result of independent (bond) percolation on $\mathscr{H}(v, f)$. Let p be the probability that a bond is open and P_p be the resulting probability measure on $\{0, 1\}^{E(\mathscr{H}(v, f))}$. It is known[5, 27] that for any $p \in [0, 1]$ the number of infinite open clusters is a.s. a constant taking from the values 0, 1, or ∞. Define

$$\begin{aligned} p_c &= p_c(\mathscr{H}(v, f)) \\ &= \inf\{p : P_p \text{ (there is an infinite open cluster on } \mathscr{H}(v, f)) = 1\} \\ p_u &= p_u(\mathscr{H}(v, f)) \\ &= \inf\{p : P_p \text{ (there is a unique infinite open cluster on } \mathscr{H}(v, f)) = 1\} \end{aligned}$$

It is obvious that $p_c \leqslant p_u$. Benjamini and Schramm[6] proved that $0 < p_c < p_u < 1$ for independent bond percolation on any planar transitive nonamenable graph with one end, in particular on $\mathscr{H}(v, f)$.

Proposition 1. For Ising models on the hyperbolic graph $\mathscr{H}(v, f)$,

(a) If $\beta < \ln(1/(1 - p_c))$, then $M(\beta) = 0$ and consequently the Gibbs state is unique.

(b) If $\beta > \ln((1 + p_c)/(1 - p_c))$, then $M(\beta) > 0$ and consequently there are more than one Gibbs state.

(c) If $\beta < \ln(1/(1 - p_u))$ and also $M(\beta) > 0$, then $\nu^f \neq (\nu^+ + \nu^-)/2$.

(d) If $\beta > \ln((1 + p_u)/(1 - p_u))$, then $\nu^f = (\nu^+ + \nu^-)/2$.

Noticing that $M(\beta)$ is a nondecreasing function of β, define

$$\beta_c = \inf\{\beta > 0 : M(\beta) > 0\}$$

$$\beta'_c = \inf\{\beta \geqslant \beta_c : \nu_\beta^f = (\nu_\beta^+ + \nu_\beta^-)/2\}$$

As mentioned in the introduction, it is known[22] that for Ising models on $\mathbf{Z}^d$, $\nu^f = (\nu^+ + \nu^-)/2$ for all but countably many β. Therefore $\beta_c = \beta'_c$ for Ising models on $\mathbf{Z}^d$ with $d \geqslant 2$. However for Ising models on $\mathscr{H}(v, f)$ we have the following

Corollary 1. For Ising models on the hyperbolic graph $\mathscr{H}(v, f)$,

(a) Whenever $\beta > \beta'_c$, $\nu^f_\beta = (\nu^+_\beta + \nu^-_\beta)/2$.

(b) If $\mathscr{H}(v, f)$ is self-dual, i.e., $v = f$, or if v is sufficiently large, then $0 < \beta_c < \beta'_c < \infty$.

3. PROOFS

We first state a comparison lemma between FK random cluster models and independent percolation. FK random cluster models are described by probability measures on the configuration of bond variables, $n = \{n_b\}$, which take the value 1—meaning the bond $b = \{x, y\}$ is open, or 0—meaning b is closed. For a finite $\Lambda \subset \mathscr{H}$, the free b.c. measure $\mu^f_{\Lambda, q, p}$ has bond-configuration probabilities proportional to

$$q^{C(n)} p^{O(n)} (1-p)^{|\Lambda| - O(n)}$$

where $C(n)$ denotes the number of distinct clusters defined by the bond configuration n, $|\Lambda|$ the number of bonds in Λ and $O(n)$ the number of open bonds in Λ. The "wired" b.c. measure $\mu^w_{\Lambda, q, p}$ is defined similarly except $C(n)$ is determined by regarding all the sites in Λ^c, as well as those sites in Λ which are connected to Λ^c by an open path, as connected. For $q \geqslant 1$, infinite volume measures $\mu^f_{q, p}$ and $\mu^w_{q, p}$ exist.[2, 11, 14] The $q = 1$ case is just the independent percolation model, where the probability measure is denoted by P_p. We will simply write μ^*_p for $\mu^*_{2, p}$, where $* = f$ or w. Let $p = 1 - e^{-\beta}$, then Ising models and FK random cluster models are related by the following identities:

$$M(\beta) \equiv \langle \sigma_o \rangle_{+, \beta} = \mu^w_p(o \leftrightarrow \infty) \tag{3.1}$$

$$\langle \sigma_x \sigma_y \rangle_{f, \beta} = \mu^f_p(x \leftrightarrow y) \tag{3.2}$$

where $x \leftrightarrow y$ means that x and y are in the same open cluster, and $o \leftrightarrow \infty$ means that the cluster of o is infinite. (See ref. 2 for more details.)

The FK random cluster model is related to independent percolation by Fortuin's comparison inequalities stated in Lemma 1 below. For a proof of these inequalities see refs. 2, 11, and 14. For two probability measures μ and μ', we write $\mu \preccurlyeq \mu'$ if $\mu(A) \leqslant \mu'(A)$ for every increasing event A, where A is called an increasing event if $\{n_b\} \in A$ implies $\{n'_b\} \in A$ whenever $n_b \leqslant n'_b$ for all bond b.

Lemma 1. For $q \geqslant 1$, let $\mu^*_{q,p}$ be a free or wired b.c. measure of the FK random cluster model in $\mathscr{H}$ and let P_p be the corresponding independent percolation measure. Then

$$\mu^*_{q,p} \preccurlyeq P_p \tag{3.3}$$

and

$$\mu^*_{q,p} \succcurlyeq P_{p'} \tag{3.4}$$

where $p' = p/(p+(1-p)\,q)$.

Although the lemma is stated for any $q \geqslant 1$, we only need the $q=2$ case to prove the results for Ising models.

Proof of Proposition 1. (a) Recall that $p = 1-e^{-\beta}$. If $\beta < \ln(1/(1-p_c))$, then $p < p_c$. By (3.1) and (3.3)

$$M(\beta) = \mu^w_p(o \leftrightarrow \infty) \leqslant P_p(o \leftrightarrow \infty) = 0$$

(b) If $\beta > \ln((1+p_c)/(1-p_c))$, then $p' \equiv p/(p+2(1-p)) > p_c$. By (3.4)

$$M(\beta) = \mu^w_p(o \leftrightarrow \infty) \geqslant P_{p'}(o \leftrightarrow \infty) > 0$$

(c) If $\beta < \ln(1/(1-p_u)$ then $p < p_u$. By (3.2), (3.3), and Theorem 4.1 of ref. 25, there exists two sequences of sites $x_n, y_n \in V(\mathscr{H}(v, f))$, $n = 1, 2,...,$ such that

$$\langle \sigma_{x_n} \sigma_{y_n} \rangle_{f,\beta} = \mu^f_p(x_n \leftrightarrow y_n) \leqslant P_p(x_n \leftrightarrow y_n) \to 0 \tag{3.5}$$

as $n \to \infty$. On the other hand, if $M(\beta) > 0$, then by GKS inequalities,

$$\langle \sigma_x \sigma_y \rangle_{\pm,\beta} \geqslant M(\beta)^2 > 0 \tag{3.6}$$

for any sites x and y. Equations (3.5) and (3.6) imply that $\nu^f \neq (\nu^+ + \nu^-)/2$.

(d) We first show that for independent percolation on $\mathscr{H}(v, f)$, when $p > p_u$ the unique infinite open cluster "traps" $\mathscr{H}(v, f)$ a.s. More precisely, if we color each site in the unique infinite open (bond-) cluster red and color the remaining sites of $V(\mathscr{H}(v, f))$ white, then all of the white site-clusters are finite a.s. Consider independent percolation on the dual graph

$\tilde{\mathscr{H}}(v, f)(=\mathscr{H}(f, v))$. Declare a bond of $\tilde{\mathscr{H}}(v, f)$ open if the corresponding bond of $\mathscr{H}(v, f)$ is closed. If the unique infinite open cluster in $\mathscr{H}(v, f)$ does not trap $\mathscr{H}(v, f)$, then there exists an infinite open cluster in $\tilde{\mathscr{H}}(v, f)$ and hence $1-p \geqslant p_c(\tilde{\mathscr{H}}(v, f))$. On the other hand we know from Theorem 3.8 of ref. 6 that $p_c(\tilde{\mathscr{H}}(v, f)) + p_u(\mathscr{H}(v, f)) = 1$. Therefore $p \leqslant p_u(\mathscr{H}(v, f))$, a contradiction to the assumption that $p > p_u$.

Let A be the event that there exists a unique infinite open cluster in $\mathscr{H}(v, f)$ and it traps $\mathscr{H}(v, f)$. Then A is an increasing event, and $P_p(A) = 1$ when $p > p_u$ by the preceding argument. If $\beta > \ln((1+p_u)/(1-p_u))$, then $p' = p/(p+2(1-p)) > p_u$. By (3.4)

$$\mu_p^f(A) \geqslant P_{p'}(A) = 1 \tag{3.7}$$

We now outline the proof that (3.7) implies $\nu^f = (\nu^+ + \nu^-)/2$. The detailed proof is similar to the proof of Proposition 2.1.5 of ref. 28, so it is omitted. From the FK representations of Ising models, the free b.c. Ising spin configurations can be generated as follows. Given a bond configuration according to μ_p^f, assign spin values independently and symmetrically to each bond cluster, i.e., assign with probability 1/2 one of the two values, $+1$ or -1, to all sites which belong to the same bond cluster. So all sites in a given bond cluster have the same value and the value is either $+1$ or -1 with probability 1/2. By (3.7), there exists a.s. either a unique infinite $(+)$ site cluster or a unique infinite $(-)$ site cluster which traps $\mathscr{H}(v, f)$; each case occurs with probability 1/2. The unique infinite $(+)$ (or $(-)$) site cluster which traps $\mathscr{H}(v, f)$ serves as a $(+)$ (or $(-)$) b.c. So $\nu^f = (\nu^+ + \nu^-)/2$. ∎

To prove the corollary we define the isoperimetric constant of a graph $\mathscr{H}$ as follows. Let $K \subset V(\mathscr{H})$ be a subset of sites and

$$\partial K = \{b \in E(\mathscr{H}) : \text{one and only one end-point of } b \text{ is in } K\}$$

be the boundary bond set of K. Denote by $|A|$ the cardinality number of A. The isoperimetric constant of $\mathscr{H}$ is defined by

$$i(\mathscr{H}) = \inf \left\{ \frac{|\partial K|}{|K|} : K \subset V(\mathscr{H}) \text{ is finite} \right\}$$

Proof of Corollary 1. (a) From the definition of β'_c, for any $\beta > \beta'_c$ $(\geqslant \beta_c)$, there exists $\beta_0 \in (\beta'_c, \beta)$ such that $\nu^f_{\beta_0} = (\nu^+_{\beta_0} + \nu^-_{\beta_0})/2$. So

$$\langle \sigma_x \sigma_y \rangle_{f, \beta_0} = (\langle \sigma_x \sigma_y \rangle_{+, \beta_0} + \langle \sigma_x \sigma_y \rangle_{-, \beta_0})/2 \geqslant M(\beta_0)^2 > 0 \tag{3.8}$$

for any x and y of $V(\mathscr{H})$, where the first inequality uses GKS inequalities. Let $p_0 = 1 - e^{-\beta_0}$. Then by (3.2) and (3.8)

$$\mu^f_{p_0}(x \leftrightarrow y) = \langle \sigma_x \sigma_y \rangle_{f, \beta_0} \geqslant M(\beta_0)^2 > 0 \tag{3.9}$$

for any x and y of $V(\mathscr{H})$. Set $p = 1 - e^{-\beta}$, then $p > p_0$ by the choice of β_0. By the stochastic monotonicity of μ^f_p and (3.9)

$$\mu^f_p(x \leftrightarrow y) \geqslant \mu^f_{p_0}(x \leftrightarrow y) \geqslant M(\beta_0)^2 > 0 \tag{3.10}$$

for any x and y of $V(\mathscr{H})$. From Theorem 4.1 of ref. 25, (3.10) implies that

$$\mu^f_p \text{ (there exists a unique infinite open cluster in } \mathscr{H}) = 1 \tag{3.11}$$

As in the proof of part (d) of the proposition, in order to show $\nu^f_\beta = (\nu^+_\beta + \nu^-_\beta)/2$, one only needs to show that

$$\mu^f_p \text{ (there exists a unique infinite open cluster in } \mathscr{H} \text{ and the infinite open cluster traps } \mathscr{H}) = 1 \tag{3.12}$$

From Theorem 3.7 of ref. 6, for independent bond percolation on $\mathscr{H}$, if there exists a unique infinite open cluster in $\mathscr{H}$ with probability one, then, with probability one, there is no infinite open cluster in the dual graph $\widetilde{\mathscr{H}}(v, f)$. The proof of this theorem can be extended word by word to any percolation process on $\mathscr{H}$ which is invariant under graph automorphisms and which has the "finite energy" property (see refs. 8 and 27 for the definition and discussion of the finite energy property). In particular, μ^f_p is such a percolation process. So (3.11) implies that

$$\mu^f_p \text{ (there is no infinite open cluster in } \widetilde{\mathscr{H}}(v, f)) = 1 \tag{3.13}$$

On the other hand, it is not hard to see that geometrically if the unique infinite open cluster in $\mathscr{H}$ does not trap $\mathscr{H}$, then there exists some infinite open cluster in $\widetilde{\mathscr{H}}(v, f)$. This observation, together with (3.11) and (3.13), imply (3.12). This completes the proof of part (a).

(b) From parts (a) and (b) of the proposition,

$$0 < \ln(1/(1 - p_c)) \leqslant \beta_c \leqslant \ln((1 + p_c)/(1 - p_c))$$

If $\ln((1 + p_c)/(1 - p_c)) < \ln(1/(1 - p_u))$, then

$$\beta_c \leqslant \ln((1 + p_c)/(1 - p_c)) < \ln(1/(1 - p_u)) \leqslant \beta'_c \leqslant \ln((1 + p_u)/(1 - p_u)) < \infty$$

where the third and the fourth inequality is due to part (c) and part (d) of the proposition respectively. So we only need to show that $\ln((1+p_c)/(1-p_c)) < \ln(1/(1-p_u))$, or equivalently that $2p_c/(1+p_c) < p_u$.

If $\mathscr{H}(v, f)$ is self-dual, then (recall that $p_c(\widetilde{\mathscr{H}}(v, f)) + p_u(\mathscr{H}(v, f)) = 1$) we have that $p_u(\mathscr{H}(v, f)) = 1 - p_c(\mathscr{H}(v, f))$. So we only need to show that $2p_c/(1+p_c) < 1 - p_c$, or $p_c < \sqrt{2} - 1$. It is known from ref. 5 that

$$p_c \leqslant 1/(i(\mathscr{H}(v, f)) + 1) \tag{3.14}$$

So it is sufficient to show that $i(\mathscr{H}(v, f)) > \sqrt{2}$ for $v = f (\geqslant 5)$. It is calculated in refs. 10 and 16 that $i(\mathscr{H}(v, f)) \geqslant \sqrt{5}$ when $v = f \geqslant 5$. This completes the self-dual case.

If $\mathscr{H}(v, f)$is not self-dual, define the spectral radius of $\mathscr{H}(v, f)$ by

$$\rho(\mathscr{H}(v, f)) = \limsup_{n \to \infty} (p_n(x, y))^{1/n}$$

which does not depend on x or y, where $p_n(x, y)$ denote the n-step transition probability between sites x and y, for the simple random walk on $\mathscr{H}(v, f)$. From Theorem 4 of ref. 5, $1/(v\rho(\mathscr{H}(v, f))) \leqslant p_u$, where recall v is the number of neighbors of each site. So it is sufficient to show that $2v\rho(\mathscr{H}(v, f))(p_c/(1+p_c)) < 1$. It is known from Theorem 2 of ref. 5 that $p_c < 1/(i(\mathscr{H}(v, f)) + 1)$. From Theorem 6 of Chapter 15 of ref. 24 or Theorem 2.1 of ref. 26, $v\rho(\mathscr{H}(v, f)) \leqslant \sqrt{v^2 - i^2(\mathscr{H}(v, f))}$ (The author learned this inequality and its application here from an argument of R. Schonmann reported in ref. 7). So it is sufficient to show that

$$2\sqrt{v^2 - i^2(\mathscr{H}(v, f))}\, \frac{1/(i(\mathscr{H}(v, f)) + 1)}{1 + 1/(i(\mathscr{H}(v, f)) + 1)} < 1$$

or

$$i(\mathscr{H}(v, f)) > \tfrac{2}{5}(\sqrt{5v^2 - 4} - 1) \tag{3.15}$$

It is calculated in ref. 16 that $i(\mathscr{H}(v, f)) = (v-2)\sqrt{1 - 4/((v-2)(f-2))}$, which is greater than the right hand side of (3.15) when v is sufficiently large for any $f \geqslant 3$. This completes the non-self-dual case. ∎

During the revision of this paper, the author learned that Häggström, Jonasson, and Lyons[16] and Schonmann[31] have proved, among many other results, similar result in their recent preprints.

ACKNOWLEDGMENTS

The author is grateful to Itai Benjamini and Oded Schramm for sending him ref. 6 before its completion and to Russell Lyons and Yuval Peres for explaining to him their results in refs. 3 and 25. He thanks a referee for a careful reading of the manuscript and for several suggestions on improving the presentation in the paper.

Research supported in part by NSF Grant DMS-98-03598.

REFERENCES

1. M. Aizenman, Translation invariance and instability of phase coexistence in the two dimensional Ising system, *Commun. Math. Phys.* **73**:83–94 (1980).
2. M. Aizenman, J. Chayes, L. Chayes, and C. M. Newman, Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models, *J. Stat. Phys.* **50**:1–40 (1988).
3. I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, Group-invariant percolation on graphs, *Geom. Func. Analysis* **9**:29–66 (1999).
4. I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, Critical percolation on any nonamenable group has no infinite clusters, *Ann. Probab.*, to appear.
5. I. Benjamini and O. Schramm, Percolation beyond $\mathbf{Z}^d$, many questions and a few answers, *Electronic Commun. Probab.* **1**:71–82 (1996).
6. I. Benjamini and O. Schramm, Percolation in the hyperbolic plane, preprint.
7. I. Benjamini and O. Schramm, Recent progress on percolation beyond $\mathbf{Z}^d$. http://www.wisdom.weizmann.ac.il/ schramm/papers/pyond-rep/index.html.
8. R. M. Burton and M. Keane, Density and uniqueness in percolation, *Commun. Math. Phys.* **121**:501–505 (1989).
9. R. L. Dobrushin, Gibbs state describing coexistence of phases for a three-dimensional Ising model, *Theory Probab. Appl.* **17**:582–600 (1972).
10. B. Dudo, R. Scheib, and C. C. Wu, Growth rates of some planar graphs—an application of Mathematica, in preparation.
11. C. M. Fortuin, On the random cluster model. III. The simple random cluster model, *Physica* **59**:545–570 (1972).
12. C. M. Fortuin and P. W. Kasteleyn, On the random cluster model. I. Introduction and relation to other models, *Physica* **57**:536–564 (1972).
13. G. Gallavotti and S. Miracle-Sole, Equilibrium states of the Ising model in the two phase region, *Phys. Rev. B* **5**:2555–2559 (1972).
14. G. R. Grimmett, The stochastic random-cluster process and the uniqueness of random-cluster measures, *Ann. Probab.* **23**:1461–1510 (1995).
15. G. R. Grimmett and C. M. Newman, Percolation in $\infty+1$ dimensions, in *Disorder in Physical Systems*, G. R. Grimmett and D. J. A. Welsh, eds. (Clarendon Press, Oxford, 1990), pp. 167–190.
16. O. Häggström, J. Jonasson, and R. Lyons, Explicit isoperimetric constants, phase transitions in the random-cluster and Potts models, and Bernoullicity, preprint.
17. O. Häggström and Y. Peres, Monotonicity of uniqueness for percolation on Cayley graphs: All infinite clusters are born simultaneously, *Probab. Th. Rel. Fields* **113**:273–285 (1999).
18. Y. Higuchi, On the absence of non-translation invariant Gibbs states for the two dimensional Ising model, in *Random Fields* (Esztergom, 1979), J. Fritz, J. L. Lebowitz and D. Szsz, eds. (Amsterdam, North-Holland, 1981), Vol. I, pp. 517–534.

19. J. Jonasson, The random cluster model on a general graph and a phase transition characterization of nonamenability, *Stoch. Proc. Appl.* **79**:335–354 (1999).
20. J. Jonasson and J. Steif, Amenability and phase transition in the Ising model, *J. Theor. Probab.*, to appear.
21. S. Lalley, Percolation on Fuchsian groups, *Ann. Inst. H. Poincaré, Probab. Stat.* **34**: 151–178 (1998).
22. J. L. Lebowitz, Coexistence of phases in Ising ferromagnets, *J. Stat. Phys.* **16**:463–476 (1977).
23. R. Lyons, Random walks and the growth of groups, *C.R. Acad. Sci. Paris Sér. I Math.* **320**:1361–1366 (1995).
24. R. Lyons and Y. Peres, Probability on Trees and Networks. Book in preparation, draft available at URL http://php.indiana.edu/~rdlyons.
25. R. Lyons and O. Schramm, Indistinguishability of percolation clusters, *Ann. Probab.*, to appear.
26. B. Mohar, Isoperimetric inequalities, growth, and the spectrum of graphs, *Linear Algebra Appl.* **103**:119–131 (1988).
27. C. M. Newman and L. S. Schulman, Infinite clusters in percolation models, *J. Stat. Phys.* **26**:613–628 (1981).
28. C. M. Newman and C. C. Wu, Markov fields on branching planes, *Probab. Th. Rel. Fields.* **85**:539–552 (1990).
29. R. Rietman, B. Nienhuis, and J. Oitmaa, The Ising model on hyperlattices, *J. Phys. A: Math. Gen.* **25**:6577–6592 (1992).
30. R. H. Schonmann, Stability of infinite clusters in supercritical percolation, *Probab. Th. Rel. Fields* **113**:287–300 (1999).
31. R. H. Schonmann, Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs, preprint.
32. C. M. Series and Y. G. Sinai, Ising models on the Lobachevsky plane, *Commun. Math. Phys.* **128**:63–76 (1990).
33. C. C. Wu, Ising models on hyperbolic graphs, *J. Stat. Phys.* **85**:251–259 (1997).